Noncommutative torus from Fibonacci chains via foliation

2001 J. Phys. A: Math. Gen. 34 R1

(http://iopscience.iop.org/0305-4470/34/31/201)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) R1–R19 PII: S0305-4470(01)16865-4

TOPICAL REVIEW

Noncommutative torus from Fibonacci chains via foliation

Hyeong-Chai Jeong[1], **Eunsang Kim**[2,4] **and Chang-Yeong Lee**[1,3]

[1] Department of Physics, Sejong University, Seoul 143-747, Korea
[2] Department of Industrial and Applied Mathematics, Kyungpook National University, Taegu 702-701, Korea
[3] Theory Group, Department of Physics, University of Texas, Austin, TX 78712, USA

E-mail: hcj@kunja.sejong.ac.kr, eskim@wavelet.hanyang.ac.kr and cylee@kunja.sejong.ac.kr

Received 31 August 2000, in final form 17 May 2001
Published 27 July 2001
Online at stacks.iop.org/JPhysA/34/R1

Abstract
We classify the Fibonacci chains (F-chains) by their index sequences and construct an approximately finite-dimensional (AF) C^*-algebra on the space of F-chains as Connes did on the space of Penrose tiling. The K-theory on this AF algebra suggests a connection between the noncommutative torus and the space of F-chains. A noncommutative torus, which can be regarded as the C^*-algebra of a foliation on the torus, is explicitly embedded into the AF algebra on the space of F-chains. As a counterpart of that, we obtain a relation between the space of F-chains and the leaf space of Kronecker foliation on the torus using the cut-procedure of constructing F-chains. Our embedding of the C^*-algebra of the foliation is consistent with the recent result of Landi, Lizzi, and Szabo that the C^*-algebra of noncommutative torus can be embedded into an AF algebra.

PACS numbers: 02.40.Gh, 61.44.Br

1. Introduction

Recently, noncommutative geometry (NCG) has been one of the most active areas in mathematics with increasing interest and application to physics [1–3]. Not only does it open new areas in pure mathematics but its application to physics has now reached to the very frontiers of fundamental physics, such as string and M theories [4, 5]. Many in the string/gravity circle now consider NCG as a very possible candidate for the underlying mathematical framework of quantum theory of gravity [6–8]. However, applications of NCG to physical systems have not been confined to high-energy physics. Bellissard applied NCG

[4] Current address: BK21 Mathematical Science Division, Seoul National University, Seoul 151-747, Korea.

to the quantum Hall effect and explained the Hall conductivity using the K-theory on the noncommutative algebra of functions on the Brillouin zone [9].

The quasicrystal seems to be another novel system in which we may be able to achieve a 'quantum leap' in progress when we adapt an NCG approach. Quasicrystals are new types of solids with ordered atomic arrangement but with a discrete point-group symmetry forbidden for periodic systems. The discovery of such materials in 1984 [10] has had a tremendous impact on condensed matter physics and material science. Until then, the only known ordered solid-state structures were crystals and solid states were considered either periodically ordered crystals or disordered amorphous materials. The structure of this new type of solid has been explained with Penrose tiling in which two types of tiles are arranged aperiodically [11]. One can show that the Penrose tiling lattice has the translational order and the rotational symmetry of observed quasicrystals by calculating their Fourier components [12]. Furthermore, Penrose tiling models provide clues for solving the puzzle of physical realization of such structures through the atomic interactions: that is, the question why the atoms form the complex Penrose tiling pattern rather than a regularly repeated crystal arrangement [13, 14]. However, the study of their dynamical properties, the most important secrets of quasicrystals, is still in its infancy. It may require a new analytical tool since quasicrystals defy the standard classical classification of solids. NCG may be a candidate for this. Connes has already pointed out that the space of Penrose tiling can be analysed nontrivially only with noncommutative algebra which has a quantum mechanical nature [2]. An indication of quantum nature may lie in the fact that the symmetry of Penrose tiling is not intuitively observed from its real-space lattice structure. As stressed by Rabson and Mermin [15], the symmetry of Penrose tiling is easily seen in the Fourier-transformed space through the phases of the wavefunctions in a scattering process, which hardly play any role in the classical treatise.

Connes' analysis of the Penrose tiling space is based on the scale invariance of the Penrose tilings. Using inflation (see section 2), a Penrose tiling can be identified with a sequence consisting of 0 and 1 [2, 16]. Two different sequences correspond to the same tiling if their entries differ only in a finite number of terms. When this equivalence relation is taken into account, the space of tilings is given by the quotient space obtained from the space of sequences mod out by the equivalence relation. As Connes pointed out in his book [2], one can hardly get any interesting information about this space if it is treated as an ordinary space with the classical tools. For given any two Penrose tilings, one cannot distinguish one from the other with any finite portion of them since it appears in both tilings [16]. This tells us that the topology of the space of tilings is trivial, namely that the space of tilings is equivalent to a single point. However, by treating the space of tilings as a quantum space or noncommutative space, one can find its interesting topological invariant, the dimension group—which is not trivial at all [2]. This is because a topologically trivial space cannot be described nontrivially by complex-valued functions. However, with operator-valued functions on this space one can explore the nontrivial structure of this seemingly trivial space.

The study of quasiperiodic structure using the noncommutative geometric approach was first done by Bellissard *et al* [17] in a one-dimensional (1D) case. They investigated its spectral properties and tried to construct a quantum observable algebra which plays the role of the above-mentioned operator-valued functions. However, their investigations fell short of geometric properties in the sense of Connes.

Recently, a study in the NCG framework was done by Landi *et al* [18] from the viewpoint of the noncommutative lattice which can be regarded as a finite topological approximation of a quantum physical model. They performed their investigation by studying the K-theory of the approximately finite-dimensional (AF) C^*-algebra. For the Penrose tiling case, they retrieved the Connes result.

However, to date, not much has been known about the underlying nature of the space of tilings. On the other hand, one can see a close resemblance of the K-theory result of Connes to that of the noncommutative torus. In this paper, we investigate this aspect of the tiling space. We first review Connes' K-theory on the space of Penrose tiling and the NCG of Kronecker foliation [2]. We do this with the space of Fibonacci chains (F-chains) which is isomorphic to the space of Penrose tilings (see section 2). Using the 'projection' method explained in section 4, both F-chains and Penrose tilings can be represented as points in the higher-dimensional torus. However, we choose the space of F-chains (rather than the Penrose tiling space) since its geometrical interpretation is simpler in the torus representation [19]. The relationship between the space of F-chains and the leaf space of Kronecker foliation on the torus is investigated using the cut-procedure of constructing F-chains. We explicitly embed the C^*-algebra of a foliation on the torus into the AF algebra on the space of F-chains and explain why the map from the F-chains to the leaves of foliation is surjective.

The organization of the paper is as follows. In section 2, we introduce the deflation method of obtaining F-chains and construct the index sequences of the F-chains. The equivalence relation between F-chains is defined based on their index sequences as in the case of Penrose tilings [2]. We then follow the Connes construction of the AF algebra and review the K-theory of the C^*-algebra on the space of F-chains [2]. The definition and the basic properties of Kronecker foliation are reviewed in section 3. In section 4, we 'lift' the F-chains to a two-dimensional (2D) hyperspace. This procedure naturally leads to the torus parametrization of the F-chains. We show that the torus parametrization becomes the Kronecker foliation on the 2-torus when the equivalence relation of F-chains is applied. This mapping from the space of F-chains to the leaf space of the foliation is surjective. There is one 'singular' leaf which corresponds to two different classes of F-chains. This 'singularity' is explained in terms of both the projection method and the cut-procedure of obtaining F-chains. In section 5, we extend the leaf space such that it can be isomorphic to the space of F-chains and embed the C^*-algebra of leaves of foliation into an AF algebra on this extended space. We first obtain the equivalence relations on the extended leaf space using the equivalence relation of corresponding F-chains [2] in the finite steps. This equivalence relation partitions the space to the finite intervals. The AF algebra is obtained as an inductive limit of the finite algebra on the space of the finite intervals. In our concluding remarks in section 6, we summarize our results and discuss the implication for future research in the properties of Penrose tiling.

2. Fibonacci chain and its K-theory

The F-chain is a typical example of a 1D quasiperiodic structure. An F-chain is a special infinite sequence of two segments, say, one short S-segment and one long L-segment with the following properties:

(1) Any finite part of the sequence appears infinite times but none of them are consecutively repeated more than two times.
(2) One type of segment (say S) cannot be consecutively repeated (SS is not allowed).

One way to obtain this sequence of segments is the 'deflation' method [16]. In this method, we start from a finite sub-chain of an F-chain. We then operate the substitution (deflation) rule, $S \to L$ and $L \to LS$, iteratively to build successive strings with increasing length. At any point in the chain, the type of segment (L or S) is uniquely determined by the chosen starting sequence. Figure 1 shows such successive iterations when the starting sequence is just one segment L. An infinite number of iterative deflations produce an F-chain.

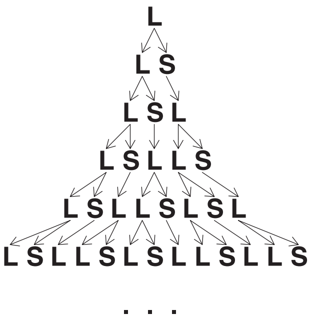

Figure 1. A way of constructing an F-chain using deflation. In each deflation step, every S segment is replaced by L and every L segment is replaced by LS. An infinite number of iterative deflations produce an F-chain.

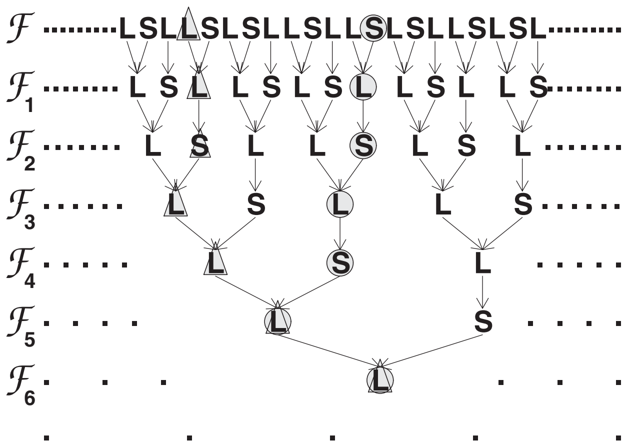

Figure 2. Successive inflations of an F-chain $\mathcal{F}$ produce a sequence of F-chains $\mathcal{F}_1$, $\mathcal{F}_2$, and so on. For the segment denoted by the triangle in $\mathcal{F}$, the index sequence is given by $(0, 0, 1, 0, 0, 0, \ldots)$ whereas that for the segment denoted by the circle is given by $(1, 0, 1, 0, 1, 0, 0, \ldots)$.

The inverse process of 'deflation' is 'inflation'. Now we begin with an F-chain $\mathcal{F}$ of two segments L and S and apply a composition (inflation), $LS \to L$ and $L \to S$. This produces another F-chain $\mathcal{F}_1$ of the two segments of L_1 and S_1, where $L_1 = LS$ and $S_1 = L$. Successive applications of the compositions yield a series of F-chains $\mathcal{F}_n$ of two segments L_n and S_n where $L_n = L_{n-1}S_{n-1}$ and $S_n = L_{n-1}$ with $L_0 := L$ and $S_0 := S$ as shown in figure 2.

This naturally introduces the index sequences of the chains [16]. For a given segment α in the original F-chain $\mathcal{F}$, the index sequence $i(\mathcal{F}, \alpha)$ is defined as an infinite sequence of integers $(a_0, a_1, a_2, \ldots)$ where $a_n = 1$ or 0 according as whether α belongs to an S_n or L_n segment in $\mathcal{F}_n$ for $n = 0, 1, \ldots$. From the inflation rule ($LS \to L$, $L \to S$), it is clear that an S segment in $\mathcal{F}_n$ must belong to an L segment in $\mathcal{F}_{n+1}$, that is, $a_n = 1$ implies $a_{n+1} = 0$ for an index sequence (a_n). In fact, one can show that the set of possible index sequences $i(\mathcal{F}, \alpha)$ of F-chains is the set Z of sequences (a_n) where $a_n \in 0, 1$ with $a_n = 1 \implies a_{n+1} = 0$ [16]. (Z is homeomorphic to the Cantor set. For details, see [31].)

Figure 2 illustrates the way of constructing the index sequence using inflation. For the segment denoted by the triangle in $\mathcal{F}$, the index sequence (a_n) is given by $(a_n) =$

$(0, 0, 1, 0, 0, 0, \ldots)$ since this segment belongs to L, L, S, L, L, L segments in the $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ chains respectively. Similarly, the index sequence (b_n) for the segment denoted by the circle is given by $(b_n) = (1, 0, 1, 0, 1, 0, 0, \ldots)$. Note that the indices in both sequences are the same for $n \geqslant 5$ since both the triangle and circle segments in $\mathcal{F}$ belongs to the same segments for $n \geqslant 5$ chains. In fact, the inflation will make any two segments in $\mathcal{F}$ separated by a finite distance belong to the same segment in $\mathcal{F}_n$ for sufficiently large n. Therefore, for the index sequences, $(a_n) = i(\mathcal{F}, \alpha)$ and $(a'_n) = i(\mathcal{F}, \alpha')$ from two given segments in the same chain, there must be an integer M such that $a'_n = a_n$ for all $n > M$. This naturally leads to the following definition of the equivalence relation $\mathcal{R}$ on Z;

$$(a_n) \sim (a'_n) \qquad \text{iff there is an integer } M > 0 \text{ such that } a_n = a'_n \text{ for all } n > M. \tag{1}$$

With this equivalence relation, it is obvious that any two index sequences from the same F-chain are in the same equivalence class.

Conversely, one can also show that any two different sequences in Z with $a_n = a'_n$ for all $n > M$ can be constructed as the index sequences from two different segments in the same F-chain [16]. Therefore, the space of F-chains is given by the quotient space $X = Z/\mathcal{R}$. In fact, this identification allows us to see the space of F-chains as a noncommutative space, as was noted in [2]. In other words, one can define the C^*-algebra associated to the quotient space X.

In what follows we review the construction of the C^*-algebra on X following the sketch of Connes [2]. Consider the set

$$Z_n = \{(a_0, \ldots, a_n) \mid a_j \in \{0, 1\} \text{ and } a_j = 1 \Longrightarrow a_{j+1} = 0\}.$$

These sets form an inverse system of sets:

$$\cdots \longrightarrow Z_{n+1} \longrightarrow Z_n \longrightarrow \cdots \longrightarrow Z_1$$

under the projection maps $Z_{n+1} \longrightarrow Z_n$ given by $(a_0, \ldots, a_n, a_{n+1}) \mapsto (a_0, \ldots, a_n)$. Note that the inverse limit $\varprojlim Z_n = Z$ is simply the set of all F-chains. On each Z_n, there is an equivalence relation $\mathcal{R}_n$ given by

$$(a_0, \ldots, a_n) \sim (a'_0, \ldots, a'_n) \qquad \text{iff} \quad a_n = a'_n. \tag{2}$$

Let $X_n = Z_n/\mathcal{R}_n$ be the set of all equivalence classes. Since each entries of sequences in Z_n are either 0 or 1, there are only two elements in X_n. Those two elements correspond to 0 or 1 in the final entry. Thus the space X_n cannot be described nontrivially by means of functions with values in complex numbers, $\mathbb{C}$. However, if we take operator-valued functions on X_n, there exists a rich class of such functions. For this, each $[x] \in X_n$, one can associate a AF Hilbert space l^2_x having elements of $[x]$ for an orthonormal basis and the algebra is given by the set of all functions on X_n with values in operators on l^2_x. Note that if the dimension of l^2_x is k, then the algebra of operators on l^2_x is the algebra of all $k \times k$ matrices. More explicitly, if x_0(x_1 resp.) represents the class in Z_n with 0 (1 resp.) in the final entry, then the dimension of $l^2_{x_0}$ ($l^2_{x_1}$ resp.) is the number of distinct elements in Z_n that end with 0 (1 resp.). Let k_n and k'_n be the dimension of $l^2_{x_0}$ and $l^2_{x_1}$, respectively. Now the algebra of functions on $[x_0]$ ($[x_1]$ resp.) with values in $M_{k_n}(\mathbb{C})$ ($M_{k'_n}(\mathbb{C})$ resp.) is simply $M_{k_n}(\mathbb{C})$ ($M_{k'_n}(\mathbb{C})$ resp.) and thus the C^*-algebra A_n of operator-valued functions on X_n is identified with $M_{k_n}(\mathbb{C}) \oplus M_{k'_n}(\mathbb{C})$. Also we have an inclusion map $A_n \longrightarrow A_{n+1}$ and it is uniquely determined by the equalities

$$\begin{pmatrix} k_{n+1} \\ k'_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_n \\ k'_n \end{pmatrix}. \tag{3}$$

It allows embedding A_n as block matrices in A_{n+1} i.e.

$$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \mapsto \begin{pmatrix} R & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & R \end{pmatrix}$$

where $R \in M_{k_n}(\mathbb{C})$ and $S \in M_{k'_n}(\mathbb{C})$. Now we have an inductive system of C^*-algebras:

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow \cdots. \tag{4}$$

Let $A = \varinjlim A_n$ be the inductive limit of the system. Then A is an AF algebra and is considered as the C^*-algebra of X. In general, approximately finite C^*-algebra or AF algebra is defined by an inductive limit of a sequence of AF C^*-algebras and such algebras can be completely classified by their K-theory [20]. Note that the algebra A described above is also known as Jones' algebra [21, 22]. The inductive system (4) can be interpreted as an infinite tower of algebras induced by the pair $A_1 \subset A_2$. From its construction, one can see that this tower is the exact counterpart of the inflation of the F-chains. Here, we present a brief review on the Jones index following Goodman [22] and explain the relationship between inflation of the F-chains and the Jones index. The following method can be used to calculate the Jones indices of the algebras associated to the spaces of general tilings which admit the inflations [23].

Let $\mathcal{L}(H)$ be the C^*-algebra of all bounded operators on a complex Hilbert space H. A *von Neumann algebra* M on H is a $*$-subalgebra of $\mathcal{L}(H)$ such that M contains the identity operator and $M = (M')'$, where $M' = \{T \in \mathcal{L}(H) \mid TS = ST \text{ for all } S \in M\}$. Note that the topology of M is different from the norm topology of the C^*-algebra $\mathcal{L}(H)$. A *factor* is a von Neumann algebra M whose centre reduces to the scalar multiples of the identity. In particular, if a factor M admits a normalized finite trace, then it is called a II_1*-factor*. Let $N \subset M$ be a pair of II_1-factors. Then the *Jones index* of N in M is defined to be

$$[M : N] = \dim_N(L^2(M)) := \mathrm{Tr}_{M'}(\mathrm{Id}_{L^2(M)})$$

where $L^2(M)$ is the Hilbert space completion of M with respect to the scalar product $\langle x|y\rangle = \mathrm{trace}(x^*y)$, for $x, y \in L^2(M)$. The trace $\mathrm{Tr}_{M'}$ is defined by $\mathrm{Tr}_{M'}(JxJ) = \mathrm{tr}_M(x)$, where $J : L^2(M) \to L^2(M)$ is the antilinear isometric involution, $J(x) = x^*$ and tr_M is the trace on M. The (*Jones*) *tower* induced by the pair $N \subset M$ is the nested sequence

$$1 \in N \subset M = M_1 \subset M_2 \subset \cdots \subset M_n \subset M_{n+1} \subset \cdots \tag{5}$$

of II_1-factors with an appropriate embedding condition. If $[M : N]$ is finite, one can always construct such a tower by taking $M_1 = JN'J$, $M_2 = JM'J, \ldots, M_k = JM'_{k-2}J$. This gives an inductive system of II_1-factors such that

$$[M_{k+1} : M_k] = [M : N].$$

It was shown in [21] that

$$[M : N] \in \left\{4\cos^2\left(\frac{\pi}{n}\right)\middle|\, n \in \mathbb{N}, n \geqslant 3\right\} \cup [4, +\infty). \tag{6}$$

In the course of the proof, Jones constructed a special type of C^*-algebra associated to the tower (5), which will be called a *Jones algebra*. The terminology is borrowed from [23]. Let M_∞ be the inductive limit of the tower (5). Then one can find a unique tracial state 'Tr' on M_∞ and a sequence of projections $e_n \in M_\infty$, i.e. $e_n = e_n^* = e_n^2$, such that:

(1) $e_n e_m = e_m e_n$ for all $n, m \in \mathbb{N}$, $|n - m| > 1$.
(2) $e_m e_{m\pm 1} e_m = [M : N]^{-1} e_m$.
(3) $\mathrm{Tr}(x e_{m+1}) = [M : N]^{-1}\mathrm{Tr}(x)$, for any x in the algebra generated by $e_1, \ldots, e_m$.

The first two conditions assert that the C^*-algebra generated by projections $e_1, \ldots, e_m$ is a AF C^*-algebra and the Jones algebra is the inductive limit of such algebras. Note that the weak closure of the resulting algebra is a Von Neumann factor of type II_1. Also the algebra depends only on $[M : N]^{-1}$. In particular, if $n = 5$, then $[M : N] = 4\cos^2\frac{\pi}{5} = (\frac{1+\sqrt{5}}{2})^2$ and the corresponding Jones algebra is simply the C^*-algebra A of the space of F-chains. In this case the trace of projections belongs to $\mathbb{Z} \oplus \mathbb{Z}$ and this leads to the study of the K-theory of an AF algebra. Recall that A is an AF algebra and is classified by its K-theory.

Now, we present an explicit calculation of K-theory of the AF algebra from the system (4) based on Connes' outline [2]. By applying basic properties of K-theory [1,20,24] to system (4), one can see that the K-theory of A is also determined by (3). Note that for each n,

$$K_i(A_n) = K_i(M_{k_n}(\mathbb{C}) \oplus M_{k'_n}(\mathbb{C})) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i = 1 \end{cases}$$

and the positive cone is given by

$$K_0^+(A_n) = \mathbb{Z}^+ \oplus \mathbb{Z}^+.$$

The map $K_0(A_n) \to K_0(A_{n+1})$ is uniquely determined by (3) and is represented by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is invertible in $\mathbb{Z} \oplus \mathbb{Z}$, it is an isomorphism on $K_0(A_n) \longrightarrow K_0(A_{n+1})$ for all $n \geqslant 0$ and we have

$$K_0(A) = \varinjlim K_0(A_n) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

On the other hand, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is not invertible in $\mathbb{Z}^+ \oplus \mathbb{Z}^+$. To compute $K_0^+(A)$, let $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ and then

$$K_0^+(A_1) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + b \geqslant 0 \text{ and } b \geqslant 0\}.$$

Now, let $\begin{pmatrix} m_n^{11} & m_n^{12} \\ m_n^{21} & m_n^{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$, then

$$K_0^+(A_n) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid m_n^{11} a + m_n^{12} b \geqslant 0 \text{ and } m_n^{21} a \geqslant 0\}.$$

Since

$$\begin{pmatrix} m_n^{11} & m_n^{12} \\ m_n^{21} & m_n^{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

with the defining relation

$$f_{n+1} = f_n + f_{n-1} \qquad \text{and} \qquad f_1 = f_2 = 1$$

we have

$$\begin{aligned} K_0^+(A_n) &= \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid f_n a + f_{n+1} b \geqslant 0 \text{ and } f_{n-1} a + f_n b \geqslant 0\} \\ &= \left\{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + \frac{f_{n+1}}{f_n} b \geqslant 0\right\}. \end{aligned}$$

From this

$$K_0^+(A) = \varinjlim K_0^+(A_n) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + \tau b \geqslant 0\}$$

where $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \tau$ is the Golden mean. Therefore, the space of F-chains is completely characterized by the ordered group

$$(K_0(A), K_0^+(A)) = (\mathbb{Z}^2, \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + \tau b \geqslant 0\}).$$

Recall that the noncommutative torus is the C^*-algebra generated by two operators u, v subject only to

$$uv = \mathrm{e}^{2\pi \mathrm{i}\Theta} vu$$

where Θ is a real number. It is well known that the K-theory of the noncommutative 2-torus A_Θ is given by $K_i(A_\Theta) \cong \mathbb{Z}^2$, where $i = 0, 1$. In particular, $K_0(A_\Theta)$ is isomorphic to $\mathbb{Z} \oplus \Theta\mathbb{Z}$ as ordered groups by a theorem of Pimsner and Voiculescu [27]. Furthermore, the noncommutative torus can be embedded into a certain type of AF algebra as discussed in Landi *et al* [28] recently. In the above, we have shown that the C^*-algebra of the space of F-chains is an AF algebra and computed its K-groups. Furthermore, the Bratteli diagram of the AF algebra satisfies the condition required by Landi *et al* 's work with $c_n = 1$ in their notation [1, 18]. Thus one might expect that the torus A_τ can be embedded into the C^*-algebra of the space of F-chains. As a dual picture, if we can realize the noncommutative torus as a geometric object, then we may characterize the space of F-chains from the space associated to the algebra A_τ. A precise relationship between the space of F-chains and the leaf space of the Kronecker foliation on the torus will be established in section 4. Before that, we first review the definition of the Kronecker foliation and study the correspondence between the noncommutative torus and the Kronecker foliation in the next section.

3. The Kronecker foliation

In general, a foliation of codimension q on an n-dimensional manifold is a partition of the manifold into p-dimensional connected submanifolds, where $n = p + q$. Such submanifolds are called the leaves of the foliation. Locally the leaves look like a set of parallel planes of codimension q in Euclidean space. The space of leaves can be understood as families of solutions of systems of differential equations and the study of foliation is the study of the global behaviour of such solutions. For example, a first-order differential equation is a vector field. For a vector field without zeros, the orbits of the flow generated by the vector field form a 1D foliation. See [25] for details on the theory of foliations.

It is well known that the 2-torus $\mathbb{T}^2$ is the only oriented compact 2-dimensional manifold which admits a non-singular codimension 1 foliation. Up to topological equivalence one can classify smooth foliations of $\mathbb{T}^2$ [25]. In particular, there is a foliation which contains no closed leaves and this foliation is equivalent to the Kronecker foliation with irrational slope. Let $\mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ with natural coordinates $(x, y) \in \mathbb{R}^2$. For non-zero constants a_1 and a_2, a smooth one-form $\omega = a_1\,\mathrm{d}x + a_2\,\mathrm{d}y$ on the torus defines a foliation on $\mathbb{T}^2$. The leaves of this foliation are the solutions of the differential equation

$$\mathrm{d}y = -\frac{a_1}{a_2}\,\mathrm{d}x.$$

If $\frac{a_1}{a_2}$ is rational, then each leaf is closed and hence a circle on the torus. If $\frac{a_1}{a_2}$ is irrational, then all the leaves are diffeomorphic to $\mathbb{R}$ and each leaf is dense in $\mathbb{T}^2$. This foliation is called the Kronecker foliation associated to a real number $-\frac{a_1}{a_2}$. From now on, we will restrict ourselves to the case when $-\frac{a_1}{a_2} = \frac{1}{\tau}$, where τ is the Golden mean. Each leaf in this case can be seen as a straight line in $\mathbb{R}^2$ with the fixed slope, $y = \frac{1}{\tau}x + b$. Since a straight line $y = \frac{1}{\tau}x + b$ is determined by its y-intercept, we see that the space of leaves of the foliation is parametrized by the y-intercepts. On the torus, two lines $\frac{1}{\tau}x + b$ and $\frac{1}{\tau}x + b'$ represent the same leaf if $b - b' = \frac{1}{\tau}n$, for some integer n. This defines an equivalence relation on the y-intercepts and the leaf space $X_{\mathcal{F}}$ of the Kronecker foliation can be identified with the set of equivalence classes. The topology on the space of leaves is the same as the quotient topology of $S^1 = \mathbb{R}/\mathbb{Z}$ divided by the partition into orbits of the rotation given by $z \mapsto z + \frac{1}{\tau}$, where $z \in S^1$, and hence there are no open sets in $X_{\mathcal{F}}$ except $\emptyset$ and $X_{\mathcal{F}}$. Therefore, the leaf space has the trivial topology as in the case of the space of F-chains.

The Kronecker foliation can be obtained also from the *suspension of diffeomorphisms* [25]. Let $\psi_\tau : S^1 \to S^1$ be the diffeomorphism which is the rotation through angle $\frac{2\pi}{\tau}$, i.e. $\psi_\tau(z) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\tau}} \cdot z,\ z \in S^1$. The product manifold $S^1 \times \mathbb{R}$ is foliated by the leaves of the form $\{z\} \times \mathbb{R}$. This foliation on $S^1 \times \mathbb{R}$ is invariant under the $\mathbb{Z}$-action on $S^1 \times \mathbb{R}$: for $z \in S^1$, $b \in \mathbb{R}$,

$$(z, b)^n = (\psi_\tau^n(z), b + n) \qquad n \in \mathbb{Z}. \tag{7}$$

This means that the quotient $S^1 \times_{\mathbb{Z}} \mathbb{R} \cong \mathbb{T}^2$ carries a 1D foliation whose leaves are the image of $\{z\} \times \mathbb{R}$ under the quotient map $S^1 \times \mathbb{R} \to \mathbb{T}^2$. Equivalently, the leaves are transverse to the fibres of $S^1 \times_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$. Thus the space of leaves of the foliation on $\mathbb{T}^2$ are parametrized by $\mathbb{R}$ together with the $\mathbb{Z}$-action associated to the action of equation (7). This is exactly the same relation as the one in the Kronecker foliation and hence the foliation obtained via the diffeomorphism ψ_τ is equivalent to the Kronecker foliation on $\mathbb{T}^2$ with the irrational slope $\frac{1}{\tau}$. This is in fact followed by the Denjoy's theorem which asserts that if a foliation of $\mathbb{T}^2$ does not have compact leaves, then it is topologically equivalent to a foliation obtained by a suspension of an irrational rotation of the circle. Also, Denjoy constructed examples of foliations on $\mathbb{T}^2$ with exceptional minimal sets and this motivated the study of minimal sets of foliations of codimension one on compact manifolds of dimension $\geqslant 3$. Here we briefly review Denjoy's example which is obtained by suspending the diffeomorphism $\psi_\tau : S^1 \to S^1$ with an exceptional minimal set [26]. In section 4, we will identify the set of all F-chains with the exceptional minimal set. A minimal set $E \subset S^1$ is the 'smallest' nonempty closed subset among the invariant subsets under ψ_τ. That is, if $E' \subset E$ is closed and invariant subset then either $E' = \emptyset$ or $E' = E$. A minimal set is called *exceptional* if it is homeomorphic to a subset of the Cantor set on S^1. The exceptional minimal set for ψ_τ is constructed in the following manner. First, cut the circle S^1 at all the points of an orbit $\{\theta_n \mid n \in \mathbb{Z}\}$ of the given irrational rotation. At the n cutting point, insert a segment J_n of length l_n with $\sum l_n < \infty$. Then we get a new circle and the set $S^1 - \cup_{n\in\mathbb{Z}} J_n = E$ is homeomorphic to the Cantor set and this is the desired exceptional minimal set.

From the above construction of the Kronecker foliation, one can relate the C^*-algebra of the foliation to the noncommutative torus. Let $C(S^1)$ be the C^*-algebra of continuous functions on S^1. Then the rotation $\psi_\tau : S^1 \to S^1$ induces the automorphism $\psi_\tau^* : C(S^1) \to C(S^1)$ given by $\psi_\tau^*(f) = f \circ \psi_\tau$, where $f \in C(S^1)$, or

$$(\psi_\tau^* f)(z) = (f \circ \psi_\tau)(z) = f(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\tau}} \cdot z) \qquad z \in S^1.$$

Let us denote the group of all automorphisms of $C(S^1)$ by $\mathrm{Aut}(C(S^1))$. Then the action of equation (7) can be given as the group homomorphism $\alpha : \mathbb{Z} \to \mathrm{Aut}(C(S^1))$ which is given by

$$(\alpha(n)f)(z) := (\alpha_n f)(z) = (f \circ \psi_\tau^n)(z) = f(\mathrm{e}^{\frac{2\pi n \mathrm{i}}{\tau}} \cdot z).$$

Now the C^*-algebra of this action is the so-called *crossed product* C^*-algebra $C(S^1) \rtimes_\alpha \mathbb{Z}$ [20]. As we have seen above, this algebra is generated by the rotation and the $\mathbb{Z}$-action. More explicitly, the C^*-algebra is represented on $L^2(S^1)$ with generators U and V according to the rotation and $\mathbb{Z}$-action:

$$(Uf)(z) = zf(z) \qquad \text{and} \qquad (Vf)(z) = f(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\tau}} \cdot z) \qquad f \in L^2(S^1) \quad z \in S^1.$$

It is easy to verify that the operators U and V satisfy the relation

$$UV = \mathrm{e}^{\frac{2\pi n \mathrm{i}}{\tau}} VU.$$

Hence the C^*-algebra $C(S^1) \rtimes_\alpha \mathbb{Z}$ is identified with the noncommutative torus $A_{\frac{1}{\tau}}$ and can be regarded as the C^*-algebra on the leaf space of the Kronecker foliation with the slope $\frac{1}{\tau}$. Also, this is Morita equivalent to the noncommutative torus A_τ [2] and hence its K-theory is given by $K_i(A_\tau) \cong \mathbb{Z}^2$, where $i = 0, 1$. In particular, $K_0(A_\tau)$ is isomorphic to $\mathbb{Z} \oplus \tau\mathbb{Z}$ as ordered groups as discussed in section 2.

4. Torus representation and the Kronecker foliation

We will establish a relation between the space of the F-chains and the leaf space of the Kronecker foliation on the torus $\mathbb{T}^2$ appearing in the torus representation [19].

An F-chain can be represented as a point in a 2-torus $\mathbb{T}^2$ [19]. Here we will show that all F-chains in an equivalent class can be represented as a leaf of the foliation on the torus. First we construct a Fibonacci lattice (F-lattice) from an F-chain. An F-lattice is a 1D tiling consists of two types of tiles L and S whose arrangements form an F-chain. The ratio of the lengths of the two tiles, $m(L)$ and $m(S)$ is given by $\frac{m(L)}{m(S)} = \tau$. Figure 3 shows an F-lattice (upper part of (*a*)) and a way of lifting it into a 2D hyperspace which is a direct product of two 1D spaces; the 'parallel space' $\mathbb{R}_\parallel$ and the 'perp-space' $\mathbb{R}_\perp$. The parallel space $\mathbb{R}_\parallel$ is a straight line parallel to the F-lattice. The perp-space $\mathbb{R}_\perp$ is the 1D space perpendicular to $\mathbb{R}_\parallel$.

The coordinate of a vertex (F-lattice point; the boundary between two given consecutive tiles) relative to any reference vertex can be expressed in the form $n_L\, m(L) + n_S\, m(S)$, where $(n_L, n_S) \in \mathbb{Z}^2$, $m(S) = \sin\theta$ and $m(L) = \cos\theta$ with $\theta = \arctan(1/\tau)$. Therefore, the vertices can be lifted into a square lattice of $\mathbb{Z}^2$ as shown in the lower part of figure 3(*a*). All pairs of adjacent vertices in the 1D tiling separated by a tile L or S are mapped onto neighbouring vertices of a 2D square lattice in the x or the y directions respectively where the x-axis has the slope $-1/\tau$. The embedded step (thick solid line) in the 2D lattice by this lifting can be covered by a strip parallel to the $\mathbb{R}_\parallel$ with width $\Delta = \cos\theta + \sin\theta$ if the position of the strip is well chosen.

For a given (infinite) F-chain, the 'perpendicular space' $\mathbb{R}_\perp$ coordinate $t_\perp$ of the strip (defined as the $\mathbb{R}_\perp$ coordinate of the bottom boundary of the strip), which covers the embedded step completely, is uniquely determined[5]. Therefore, we can assign a $t_\perp$ value for a given F-chain. The $\mathbb{R}_\parallel$ coordinate $t_\parallel$ of the chain is not uniquely determined but depends on the choice of the vertex in the 1D tiling. Figure 3(*a*) shows two different values: $t_{1,\parallel}$ for the triangle vertex and $t_{2,\parallel}$ for the circle vertex. In section 2, we mentioned that two sequences in Z which are equivalent by $\mathcal{R}$ of equation (1) can be constructed from two different segments in the same F-chain. In other words, two F-chains in an equivalence class can be considered as a finite translation of each other. Since the translation in the $\mathbb{R}_\parallel$ direction corresponds to the movement along the leaf in the torus representation, all F-chains in an equivalent class can be represented as the points on the same leaf on the torus no matter what vertices we choose for $t_\parallel$.

Conversely, an F-lattice (hence an F-chain) can be obtained from a 2D square lattice by the projection methods. The lattice sites of the 2D square structure can be projected onto the 1D parallel space, $\mathbb{R}_\parallel$ at the slope $\tan\theta = 1/\tau$ with respect to the horizontal rows of the square lattices. Since the slope of the line is irrational, the projection of all 2D lattice points to $\mathbb{R}_\parallel$ form a dense set of points. If we restrict projections on $\mathbb{R}_\parallel$ to the points confined within a strip which is parallel to $\mathbb{R}_\parallel$ and has a cross section Δ in $\mathbb{R}_\perp$ equal to the perp-space projection of a square unit cell ($\Delta = \cos\theta + \sin\theta$), then the projection to $\mathbb{R}_\parallel$ gives an F-lattice [29]. The

[5] In the upper part of figure 3(*a*), only a finite part of the F-chain is shown. Therefore, the position of the strip which covers the finite embedded step is not uniquely determined. All three strips in the figure cover the finite embedded step. The highest strip shown in (*b*), whose lower boundary passes the lattice point, corresponds to the infinite F-chain whose index sequence is (101010...) while the lowest strip in (*c*) corresponds to (01010...). Details will be discussed later.

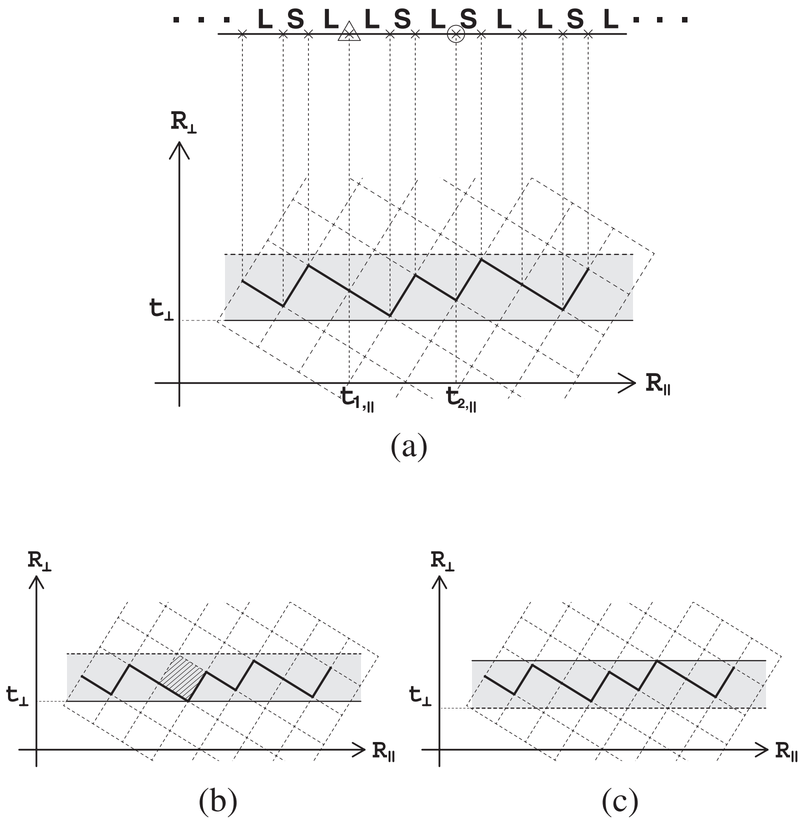

Figure 3. The lifting of a Fibonacci chain (F-chain) into a 2D hyperspace. For a given F-chain, we can construct a 1D tiling consists of two types of tiles L and S which can be lifted into a 2D square lattice whose x-axis has the slope $-1/\tau$ with respect to the 1D tiling on $\mathbb{R}_\parallel$. The embedded step (solid thick line) in the 2D lattice can be covered by a strip parallel to $\mathbb{R}_\parallel$ with width $\Delta = \cos\theta + \sin\theta$ when the position of the strip is well chosen. The perp coordinate $t_\perp$ is given by the $\mathbb{R}_\perp$ coordinate of the strip bottom. This value is uniquely determined for a given infinite F-chain. In the upper part of (a), only a finite part of the F-chain is shown. Therefore, the position of the strip which covers the finite embedded step is not uniquely determined. All three strips in (a)–(c) cover the shown finite part but correspond to different (infinite) F-chains. The highest and the lowest strips, shown in (b) and (c) respectively, correspond to the singular F-chains. The parallel coordinate $t_\parallel$ of the F-chain depends on the choice of the vertex in the tiling.

movement of the strip along the perp-space $\mathbb{R}_\perp$ gives rise to rearrangement of tiles from one perfect F-lattice to another and the strip is called the 'window' of the corresponding F-lattice. In general, the windows should include one and only one boundary to produce a perfect F-lattice. Figure 3 can be also used to illustrate the 'projection methods'. Now we first choose a window and select the 2D lattice points which are in the window. Then the projection of those lattice points into the $\mathbb{R}_\parallel$ space gives the vertices of an F-lattice. The boundaries are irrelevant to the projected structure unless they pass a 2D lattice point since all vertices of the F-lattice are produced from the 2D lattice points inside the window (figure 3(a)). When a boundary of the window intersects with a 2D lattice point, so does the other boundary as shown in figure 3(b) since the width of the window is equal to the perp-space projection of a square unit cell. If the window included both boundaries, the projection would produce an extra vertex from the 2D unit cell denoted by hatching. On the other hands, if it excluded both,

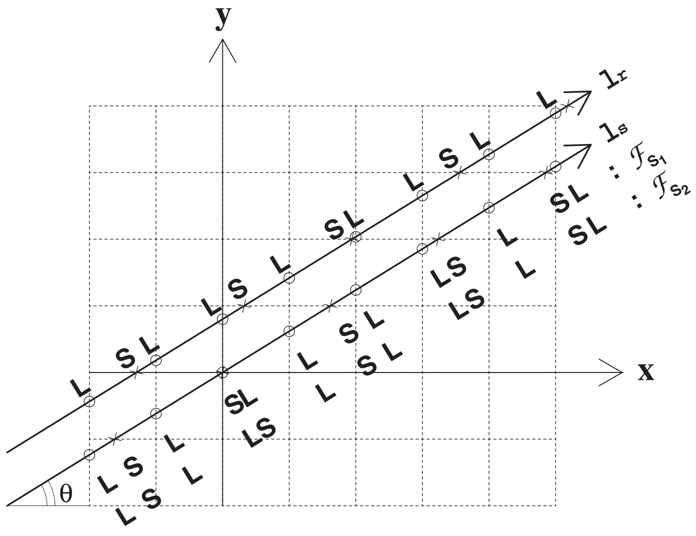

Figure 4. An F-chain can be obtained from the sequence of intersections between a line with slop $1/\tau$ and the x and y axes. A regular line l_r intersects $\ldots, y, x, y, y, x, y, x, y, y, x, y, \ldots$ axes and corresponds to a unique F-chain, $\ldots LSLLSLSLLSLL \ldots$. The singular line l_s, which passes through the origin, corresponds to the two different F-chains, $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$.

the projected lattice would miss a vertex. Therefore, the 'proper' windows must include one and only one boundary. If we include the lower boundary as shown in figure 3(*b*), then we get the tile arrangement 'LS' from the hatched unit cell, while we get the 'SL' arrangement for the other case. In other words, there are two different F-lattices (and hence two different F-chains) corresponding two different proper windows in spite of the $\mathbb{R}_\perp$ position of the window is the same.

This 'singularity' for the windows whose boundaries pass the lattice points can be more clearly understood in the cut-procedure. A leaf on the torus can produce an F-chains (but not an F-lattice) directly (instead of going through a strip or a window) in the cut-procedure. Figure 4 illustrates a way to get an F-chain by this method from a square lattice in a 2D 'hyperspace'. We consider a 2D square lattice and the lines with the slope of $1/\tau$. (A straight line in a square lattice can be considered as a representation of a leaf on the torus in the 'extended' scheme.) We can produce an F-chain associated to the line in the following way. If the line intersects the y-axis, we give the segment 'L' while we assign the segment 'S' when the line intersects with the x-axis. For example, the line l_r in figure 4 intersects $\ldots, y, x, y, y, x, y, x, y, y, x, y, \ldots$ axes and hence produces an F-chain $\mathcal{F}_r = \ldots LSLLSLSLLSLL \ldots$. The correspondence between a straight line and an F-chain is one-to-one except for the 'singular' line l_s which passes through the origin. For the singular case, the three lines, the parallel line l_s, the x-axis and the y-axis meet at a point (at the origin). Since l_s meets both the x and the y axes at the same time, a pair of segments (one S and one L segments) should be assigned at the origin. However, assigning two segments at the same point is impossible. This 'singularity' can be resolved by moving the line l_s infinitesimally. If we move l_s slightly upward, it first meets the x-axis and then the y-axis and 'SL' is assigned at the origin. Therefore, the parallel line l_s produces the F-chain $\mathcal{F}_{s_1}$ in figure 4 when it is moved upward infinitesimally. In contrast, if we move l_s slightly downward, it meets the y-axis first and then meets the x-axis. Therefore, 'LS' is assigned at the origin and we have $\mathcal{F}_{s_2}$ in this case.

Now, the space of F-chains can be parametrized by the y-axis in $\mathbb{R}$ since the line $y = \frac{1}{\tau}x+b$ is determined by the y-intercepts. As in the leaf space of the Kronecker foliation, two F-chains that correspond to two lines $\frac{1}{\tau}x + b$ and $\frac{1}{\tau}x + b'$ are in the same equivalence class if $b - b' = n/\tau$. This is because the arrangements of intersections from the two lines, hence the two corresponding F-chains, are the same up to the finite translation (by $n\sqrt{\tau^2 + 1}/\tau$) when $b - b' = n/\tau$. If two F-chains $\mathcal{F}$ and $\mathcal{F}'$ are the same up to a finite translation, they are in the same class by the equivalence relation $\mathcal{R}$ of equation (1). Let (a_n) and (a'_n) be index sequences of the two F-chains with $(a_n) = i(\mathcal{F}, \alpha)$ and $(a'_n) = i(\mathcal{F}', \alpha')$ where α and α' are the segments at the origins of $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then, there is a segment $\alpha'' \in \mathcal{F}$ within a finite distance from the origin such that $(a''_n) = i(\mathcal{F}, \alpha'')$ is identical to (a'_n) since $\mathcal{F}'$ is a finite translation of $\mathcal{F}$. Now, we have an integer M such that $a_n = a'_n$ for all $n > M$ since the inflation will make two segments α and α'' belong to the same segment in $\mathcal{F}_n$ for sufficiently large n.

We have shown that the space of F-chains is the same as the leaf space of Kronecker foliation except for the singular leaf which corresponds to the two F-chains. One may think that the two F-chains corresponding to the singular leaf are of the same class. The only difference between the two chains is at the origin, and can be removed by local surgery; we can obtain one chain from the other by flipping a pair of segments at the origin. Furthermore, one is a mirror image of the other and related by a 180° rotation. However, they are not in the same class by the equivalence relation $\mathcal{R}$ of equation (1). If we construct the index sequences of the two F-chains from the segment at the origin, one chain corresponds to the index sequence $(a_n) = (0101010101\ldots)$ and the other corresponds to $(a'_n) = (1010101010\ldots)$ (see section 4). In other words, $a_n = \delta_{n,2k}$ for one chain while $a'_n = \delta_{n,2k+1}$ for the other chain. Clearly a_n and a'_n are different for all n and they are in different classes.

Since we have two different F-chains on the singular leaf (which is only one leaf on the Kronecker foliation) on $\mathbb{T}^2$, we have a surjective map from the space of F-chains to the space of leaves. Both spaces have trivial topology and the map is open and continuous. Now, the surjectiveness corresponds to the fact that the map from the C^*-algebra of leaves of foliation (noncommutative torus) to the C^*-algebra of F-chains (we already showed in section 2 that it is an AF algebra) is injective. In this sense our discussion above can be seen as a dual picture of the embedding of noncommutative torus into a certain type of AF algebra.

In the following section, we construct such an AF algebra by introducing an extended space of leaves which is isomorphic to the space of F-chains.

5. An extended space of leaves

Since the map from the space of F-chains to the leaf space of the Kronecker foliation is surjective, we cannot retrieve all F-chains from the leaves on $\mathbb{T}^2$. However, there is only one leaf which corresponds to more than one class of F-chains. Furthermore, this singular leaf corresponds to only two classes of F-chains. Therefore, if we assign one class of F-chain to every leaf (including the singular leaf), all F-chains except only one class of F-chains are obtained from the leaf space. For example, if we assign $\mathcal{F}_{s_2}$ in figure 4 to the singular leaf, then $\mathcal{F}_{s_1}$-class is 'missing' but all other F-chains are in the leaf space on $\mathbb{T}^2$. In this section, we show that an extended leaf space, which is isomorphic to the space of F-chains, is naturally obtained if we construct the equivalence relations on the leaf space using the equivalence relation $\mathcal{R}_n$ of the finite subsequences of the index sequences given by equation (2). In the limit of the length of the subsequences goes to infinity, we get the extended leaf space which is the sum of two spaces; the leaf space on $\mathbb{T}^2$ and the space consists of one leaf corresponding to the 'missing' class.

In section 2, the index subsequences of the F-chains are constructed using the inflation and the AF algebra is introduced as an inductive limit of the finite algebra on the space of the finite subsequences of the index sequences. There were only two sets of equivalence classes on the space of the finite subsequences since the last entries of sequences a_n in Z_n can have only two values, either 0 or 1. Here, we consider the space of straight lines in the cut-procedure of figure 4. Except for the singular lines which pass the lattice points, each line produces one and only one F-chain. The equivalence relation between the lines is constructed according to the equivalence relation $\mathcal{R}_n$ for their F-chains (equation (2)). As before, the straight lines are parametrized by the y-intercepts. To define the equivalence relation between the lines, we consider the index sequences of their F-chains $i(\mathcal{F}(b), \alpha)$ where α is the segment at $x = 0$ (at the y-axis) and $\mathcal{F}(b)$ is the F-chain corresponds to the line $y = \frac{1}{\tau}x + b$. The index a_0 of the original (uninflated) F-chain $\mathcal{F}$ is 0 for every $b \in W_0 := (0, 1)$ since the type of the segment at the origin is always L by the definition of the cut-procedure (at $x = 0$, the leaf $y = \frac{1}{\tau}x + b$ always cut the y-axis). Figure 5(a) illustrates this: any $b \in W_0$ corresponds to the L segment in $\mathcal{F}$ at the origin. However, the type of segment at the origin in the inflated F-chains $\mathcal{F}_1$ can be both L and S depending on the value of b. As shown in figure 5(b), the segment arrangement in $\mathcal{F}$ at the origin is LS for $b \in A = (1/\tau^2, 1)$, and LL for $b \in B = (0, 1/\tau^2)$. Since LS becomes L and L becomes S by inflation, the segment type in $\mathcal{F}_1$ at the origin is L for $b \in A$ and S for $b \in B$. Therefore, W_0 is partitioned by two open intervals, A and B, and a boundary point $b_1 = 1/\tau^2$ for $n = 1$. Let us denote the union of the two open intervals by W_1; $W_1 = (0, 1/\tau^2) \cup (1/\tau^2, 1) = W_0 - b_1$. Note that the boundary point b_1 is given by the intersection between W_0 and the line which passes the 2D lattice point (1, 1). For this boundary line $y = \frac{1}{\tau}x + (1 - 1/\tau)$, the segment type at the origin of $\mathcal{F}_1$ is not well defined, that is, b_1 is the singular point for $\mathcal{F}_1$. Similarly, we can partition W_1 by considering the doubly inflated F-chains $\mathcal{F}_2$. Since an S-segment in $\mathcal{F}_1$ becomes an L-segment in $\mathcal{F}_2$, the interval B in figure 5(b) is not divided for $n = 2$ (interval C in figure 5(c)). The interval A in figure 5(b) which represents the class of $a_1 = 0$ is divided by two intervals by the line which passes the 2D lattice point (2, 2). Therefore, we get the three intervals, $W_2 = (0, 1/\tau^2) \cup (1/\tau^2, 2/\tau^2) \cup (2/\tau^2, 1)$ for $n = 2$ as shown in figure 5(c). In general, an interval corresponding to $a_n = 1$ becomes an interval corresponding to $a_{n+1} = 0$ in the next step while an interval corresponding to $a_n = 0$ will be divided as two neighbouring intervals, one for $a_{n+1} = 0$ and the other for $a_{n+1} = 1$. Therefore, the partitioned interval for the nth inflated chain, which will be denoted by W_n, is given by the union of f_{n+2} intervals with f_{n+1} L-intervals and f_n S-intervals:

$$W_n = W_{n,L} + W_{n,S}$$

where

$$W_{n,L} = \sum_{k=1}^{f_{n+1}} I_{n,L_k} \qquad W_{n,S} = \sum_{k=1}^{f_n} I_{n,S_k}.$$

Here, I_{n,L_k} (I_{n,S_k} resp.) is the kth interval of L-type (S-type resp.) in W_n and f_k is the Fibonacci number introduced in section 2. The lengths of the L and the S-intervals in W_n are $1/\tau^n$ and $1/\tau^{n+1}$ respectively.

Figure 6 shows the arrangement of I_{n,L_k} and I_{n,S_k} in W_n. From figure 5, we see that the intervals are divided by the lines which pass the 2D lattice point (r, s) such that $0 < s - r/\tau < 1$. Let us arrange such lattice points according to the 'parallel' distance $d(r, s) = \frac{\tau}{\tau+2}(r + s/\tau)$ and denote them as $P_k = (r_k, s_k)$ where $d(r_{k'}, s_{k'}) < d(r_k, s_k)$ for $k' < k$. Now, let l_k be the line which passes the lattice point P_k. Then the f_{n+1} lines, $l_{f_{n+2}}, \ldots, l_{f_{n+3}-1}$, divide the f_{n+1} L-intervals in W_n. For example, l_1 which passes $P_1 = (1, 1)$ divides the L-interval in W_0, l_2

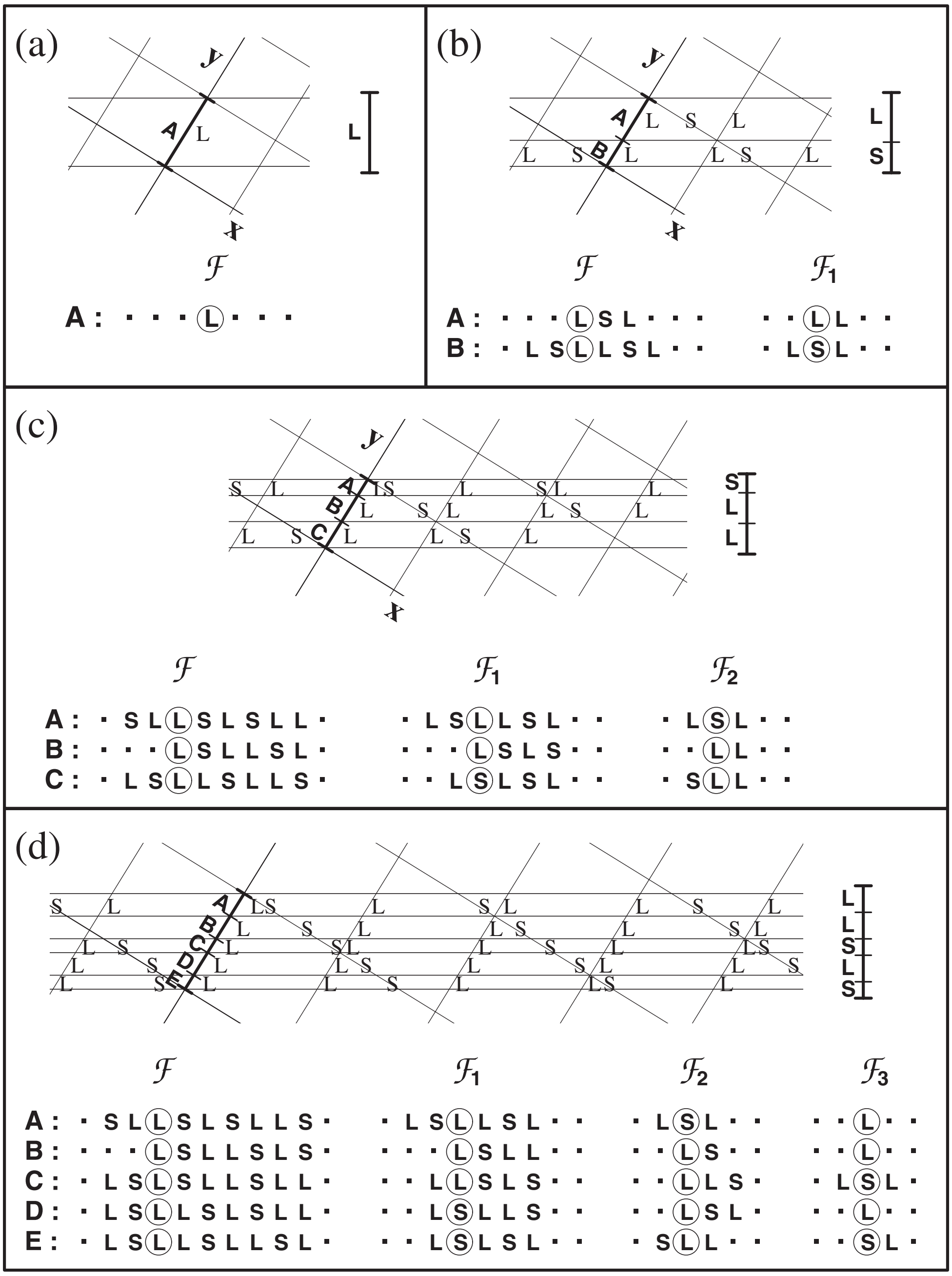

Figure 5. A sequence of partitions of the transversal (1, 0) using the cut-procedure and the inflations of the F-chain. $\mathcal{F}_{n+1}$ is the inflation of $\mathcal{F}_n$ as in figure 2.

which passes $P_2 = (2, 2)$ divides the L-interval in W_1, and l_3 and l_4 which pass $P_3 = (3, 2)$ and $P_4 = (4, 3)$ divide the L-intervals in W_2. Note that l_1 divides the L-interval of W_0 such that the lower part of it becomes an S-interval in W_1 while l_2 divides the L-interval in W_1 such

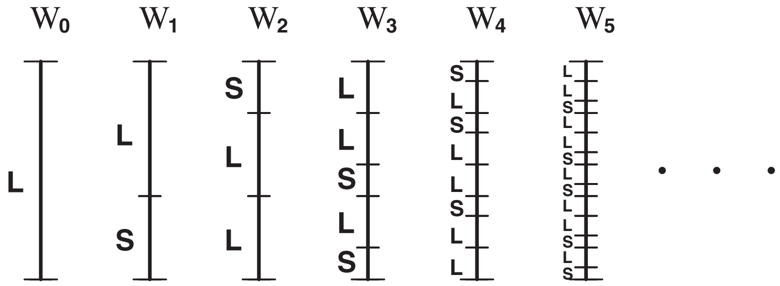

Figure 6. A sequence of partitions of the transversal for the construction of a sequence of AF algebras. An L-interval in W_n is divided by an L- and an S-interval in W_{n+1} while an S-interval becomes an L-interval without partition in the next step. For even n, the lower part of an L-interval becomes an S-interval in the $(n+1)$th inflation while the upper part becomes an S-interval for odd n.

that the upper part of it becomes an S-interval in W_2. The L-intervals in W_2 are divided by the lines l_3 and l_4 as the way that the L-interval of W_0 was divided. In fact, the lower part of an L-interval becomes an S-interval for even n while the upper part of an L-interval becomes an S-interval for odd n by an inflation[6]. Successive application of these processes produces the sequence of partitions shown in figure 6 and the two groups of intervals $W_{n,L}$ and $W_{n,S}$ can be obtained for every n.

From the construction above, we can see that the set of all intervals in W_n is isomorphic to Z_n in section 2. For a given sequence $z_n = (a_1, a_2, \ldots, a_n)$ in Z_n, we can choose an interval in W_n in the following manner. First, choose the interval L (S resp.) in W_1 if $a_1 = 0$ ($a_1 = 1$ resp.). Then choose the interval L (S resp.) in W_2, which is a subinterval of the previously chosen one if $a_2 = 0$ ($a_2 = 1$ resp.). For $a_n = 0$ ($a_n = 1$ resp.), choose the interval L (S resp.) which is a subinterval of the chosen interval in W_{n-1}. Then, there is always an interval in W_n for a given sequence in Z_n since an interval S in W_k becomes an interval L in W_{k+1}. Conversely, an interval in W_n can be indexed by a sequence in Z_n by recording the types of the intervals in W_k (for $k = 1, \ldots, n$) which the chosen interval belongs to. Now we can identify $W := \lim_{n\to\infty} W_n$ with the set $Z = \lim_{n\to\infty} Z_n$ and hence W is the set of all F-chains. Note that the boundary points excluded from the nth partitioned interval, $W_0 - W_n$, are the first f_{n+1} orbit points of the irrational rotation $-1/\tau$ from $1 - 1/\tau = 1/\tau^2$. In other words, the lines through lattice points correspond to the orbit of the rotation defined by the diffeomorphism

[6] This can be shown by two steps. (1) All L-intervals in W_n are divided in the same pattern. (2) A particular L-interval in W_n is divided as in the way mentioned in the text. To prove (1), let l_k and $l_{k'}$ be the two boundary lines of an L-interval in W_n and pass the lattice points $P_k = (r_k, s_k)$ and $P_{k'} = (r_{k'}, s_{k'})$ respectively. The irrationality of the slope guarantees the same $|\Delta r| = |r_k - r_{k'}|$ and $|\Delta s| = |s_k - s_{k'}|$ for all L-intervals in W_n; all of them are given by $|\Delta r| = f_n$ and $|\Delta s| = f_{n-1}$ due to the relations $f_{n-1} - f_n/\tau = (-1)^n \frac{1}{\tau^n}$. Now, the L-intervals in W_n can be mapped to pairs of 2D lattice points which are identical up to a lattice translation. In other words, relative arrangements of the lattice points from the 'boundary' lattice points are the same for all L-intervals in W_n and hence divided in the same pattern by the boundary lines in the following inflations. The statement (2) can be shown with the bottom-most interval for even n and the topmost interval for odd n with an inductive way. Here we outline the proof for even n. The odd n case can be proven similarly. For $n = 2$, the bottom-most interval is an L-interval and the upper boundary is given by the line which passes the lattice point $(1, 1)$. Let the bottom-most interval of the $n = 2k$ case be an L-interval. Then the size of the interval is τ^{-n} and the upper boundary line passes the lattice point (f_n, f_{n-1}). This interval will be divided by the lines which pass the lattice points (r, s) such that $0 < r - s/\tau < \tau^{-n}$. Since f_n/f_{n-1} is the 'best' rational approximant of τ [30], the lattice point (f_{n+2}, f_{n+1}) has the smallest parallel distance and hence the line which pass it divides the interval first. The 'perpendicular' distance of this lattice point is $f_{n+1} - f_{n+2}/\tau = \tau^{-(n+2)}$ and we see that the lower partition of the interval becomes an S-interval in W_{n+1}. Furthermore, this implies that the bottom-most interval for $n + 2$ is L-type and the above argument can be applied inductively for all even n.

ψ_τ introduced in section 3. In fact the construction of W is exactly the same as that of the exceptional minimal set for the suspension of diffeomorphism ψ_τ. Thus the set W or the set of all F-chains is the exceptional minimal set and also the set Z is homeomorphic to the Cantor set as asserted in [2, 31].

Now, let us give an equivalence relation $\tilde{\mathcal{R}}_n$ on W_n as $\mathcal{R}_n$ of equation (2) on Z_n. Then the set of equivalence classes $\tilde{X}_n = W_n/\tilde{\mathcal{R}}_n$ has only two elements, $W_{n,L}$ and $W_{n,S}$, which have f_{n+1} and f_n intervals respectively. By taking these f_{n+1} and f_n intervals as the bases of the $W_{n,L}$ and $W_{n,S}$ classes respectively, we recover the sequence of finite algebras described in section 2. In the limit of n goes to infinity, we obtain an AF algebra which is the same as the algebra in section 2.

We now show that the space $\tilde{X} = \lim_{n\to\infty} \tilde{X}_n$, which is isomorphic to the space of F-chains X, is given by the quotient space obtained from W mod out by the 'leaf equivalence relation': $b \sim b'$ iff $b - b' = n/\tau$ for some integer n, and call $\tilde{X}$ as 'extended leaf space'. First, note that an S-interval in W_n is not divided but simply becomes an L-interval in W_{n+1} while an L-interval in W_n is divided into two intervals, one L-interval and one S-interval in W_{n+1}. An important consequence of the partition sequence of figure 6 is that all L-intervals in W_n are divided in the same pattern in W_{n+1} (see footnote 6 also). Therefore, all intervals of the same type in W_n are divided in the same pattern in W_m for all $m > n$. This observation provides the relation between two points, b_z and $b_{z'}$, in an equivalence class which can be indexed by two sequences $z = (a_k)$ and $z' = (a'_k)$ with $a_m = a'_m$ for all $m \geqslant n$. If I_n (I'_n resp.) is the interval in W_n, to which b_z (b'_z resp.) belongs, the relative distance from a reference point (say, the centre) of I_n to b_z is the same as that from the centre of I'_n to $b_{z'}$ because $a_m = a'_m$ for all $m \geqslant n$. Since the lengths of L-intervals and S-intervals in W_n are τ^{-n} and $\tau^{-(n+1)}$ respectively, the distance between b_z and $b_{z'}$ is given by

$$\begin{aligned} b_z - b_{z'} &= k_L \tau^{-n} + k_S \tau^{-(n+1)} \\ &= (-1)^n \left[(k_S f_{n+1} - k_L f_n)\frac{1}{\tau} + (k_L f_{n-1} - k_S f_n) \right] \\ &= \frac{k_1}{\tau} - k_2 \end{aligned} \tag{8}$$

with integers $k_1 = (-1)^n (k_S f_{n+1} - k_L f_n)$ and $k_2 = (-1)^n (k_L f_{n-1} - k_S f_n)$. Here k_L and k_S are the number of L- and S-intervals between the two chosen intervals in W_n and we used the relation $\tau^{-k} = (-1)^k (f_{k-1} - f_k/\tau)$. These are exactly the same condition for the same leaf on the torus in section 3.

We should note that the space obtained by the limit of the above procedure is not the space of leaves of the Kronecker foliation on $\mathbb{T}^2$. If we follow the very bottom intervals of W in figure 6, we get the F-chain whose index sequence is given by (010101010...) while we get (001010101...) when we follow the very top intervals. Therefore, the limit points of these two sequences, 0 and 1 represent different classes. In the foliation on the torus, above two leaves had to be identified since 0 and 1 are identical point in S_1. This was the reason that the singular leaf on the torus corresponds two distinct F-chains.

6. Concluding remarks

In this paper, we studied the space of F-chains from the perspective of NCG. We defined the equivalence relation of the F-chains based on their index sequences and the space of F-chains X is given as the quotient set $Z/\mathcal{R}$ of all F-chains Z divided by the equivalence relation $\mathcal{R}$ of equation (1). This space is exactly the same as the space of Penrose tiling considered by Connes [2]. From the calculation of its K-theory, we know that the K_0-group of Penrose tiling (and hence F-chains) is isomorphic to that of the noncommutative torus. Furthermore, an F-chain can be parametrized as a point on the torus $\mathbb{T}^2$ [19]. These facts suggest a strong connection between the noncommutative torus and the space of F-chains. However, a C^*-algebra on the space of F-chains cannot be a noncommutative torus. From Connes' work [2], we know that the construction of a nontrivial algebra on the space of F-chains gives rise to an AF algebra whose K_1 vanishes while the K_1 of the noncommutative torus does not.

Here, we studied the exact relationship between the noncommutative torus and the AF algebra on the space of F-chains. Using the torus representation and the cut-procedure, we found a surjective map from the space of F-chains to the space of leaves on Kronecker foliation. The surjectiveness of the map and the embedding of the C^*-algebra of noncommutative torus to an AF algebra was explicitly shown by considering a sequence of finite algebra constructed on the finite partitions I_{n,L_k} and I_{n,S_k}, the quotient space $W_n/\mathcal{R}_n$. In the limit of n goes to infinity, the quotient space is identified with the space of F-chains. This is a dual approach to the way of embedding of the C^*-algebra of the space of leaves on Kronecker foliation into the C^*-algebra of the space of F-chains.

We think the current method of finding the relationship between the space of F-chains and the space of leaves of Kronecker foliation can be applied to the space of Penrose tilings. As an F-chain can be represented as a leaf on $\mathbb{T}^2$, which is a line parallel to the 1D space spanned by a 2D vector $(\cos\theta, \sin\theta)$ with $\tan\theta = 1/\tau$, a Penrose tiling can be represented as a plane on $\mathbb{T}^5$ which are parallel to the 2D space spanned by two 5D vectors, $(1, c_1, c_2, c_3, c_4)$ and $(1, s_1, s_2, s_3, s_4)$ where $c_k = \cos\left(\frac{2\pi}{5}k\right)$ and $s_k = \sin\left(\frac{2\pi}{5}k\right)$ [29]. Recall that a leaf on $\mathbb{T}^2$, $y = \frac{1}{\tau}x + b$ is parametrized by the y-intercept which can be considered as the position of the origin of the leaf. Similarly, a Penrose tiling can be parametrized by the position of the origin of the plane. By introducing the equivalence relation between the positions of the planes according to the equivalence relationship of their Penrose tiling, we can construct a space of wrapping 2D plans ('2D leaf') in the 5D space. From the identity between the space of F-chains and that of Penrose tiling, one may expect that this space should be very similar to the space of leaves of Kronecker foliation. However, a preliminary study shows that this may not be the case. The properties of the singular plane in this space, which passes through the origin of the 5D space, may behave quite differently from the singular leaf of F-chains. The singular plane corresponds to five different Penrose tilings but their index sequences are the same and given by $(0, 0, 0, \ldots)$. Therefore, all of them are in the same equivalence class unlike the F-chains from the singular leaf. Further work on this issue may have practical application for the study of quasicrystal structures. The decapod defects, which can be a seed for rapid quasicrystal growth [32], are known to be related to the singularity of the plane which passes the origin of the 5D space [33]. If future studies establish the role of the higher-dimensional singular 'leaf' in the hyper lattice space for the space of general quasiperiodic tilings, they may provide a new clue to solve the old puzzle of the topological character of the decapod defects [29].

We hope the current study on the space of F-chains provides a motive for the proliferation of noncommutative geometrical approaches for the properties of Penrose tiling and quasicrystals. At the moment, the progress on the dynamical properties of the quasicrystalline structure seems to be slow. There have been many studies on the dynamical properties on the 1D quasiperiodic systems but the conventional extensions of such studies to the higher-dimensional systems have not been able to provide a general theory on the dynamics of quasicrystals. We speculate that the study on the dynamics of 1D quasiperiodic system from the noncommutative geometrical aspect may provide a new tool for the investigation to the dynamics of quasicrystals. As shown in this paper, both the space of F-chains (1D quasiperiodic systems) and the space of Penrose tiling (2D quasicrystalline lattice) show the same nontrivial structure only with a noncommutative geometrical approach.

In summary, we show that the noncommutative torus can be obtained from the space of F-chains via foliation. We hope that this understanding will help to enhance the understanding of the dynamics of the quasicrystalline structure in future.

Acknowledgment

This work was supported by Korea Research Foundation, Interdisciplinary Research Project 1998-D00001.

References

[1] Landi G 1997 An introduction to noncommutative spaces and their geometries *Preprint* hep-th/9701078
[2] Connes A 1994 *Noncommutative Geometry* (New York: Academic)
[3] Madore J 1995 *An Introduction to Noncommutative Differential Geometry and its Physical Applications* (Cambridge: Cambridge University Press)
[4] Connes A, Douglas M R and Schwarz A 1998 *J. High Energy Phys.* JHEP02(1998)003
(Connes A, Douglas M R and Schwarz A 1997 *Preprint* hep-th/9711162)
[5] Seiberg N and Witten E 1999 *J. High Energy Phys.* JHEP09(1999)032
(Seiberg N and Witten E 1999 *Preprint* hep-th/9908142)
[6] Douglas M 1999 Two lectures on D-Geometry and noncommutative geometry *Preprint* hep-th/9901146
[7] Banks T, Fischler W, Shenker S H and Susskind L 1997 *Phys. Rev.* D **55** 5112
[8] Madore J and Mourad J 1998 *J. Math. Phys.* **39** 423
(Madore J and Mourad J 1996 *Preprint* gr-qc/9607060)
[9] Bellissard J 1986 K-theory of C^*-algebras in solid state physics *Statistical Mechanics and Field Theory: Mathematical Aspects (Lecture Notes in Physics vol 257)* (Berlin: Springer)
Bellissard J 1988 Ordinary quantum Hall effect and noncommutative cohomology *Proc. on Localization in Disordered Systems (1986)* (Leipzig: Teubner)
For a review on this topic, see Bellissard J, van Elst A and Schulz-Baldes H 1994 *J. Math. Phys.* **30** 5374
[10] Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
[11] Penrose R 1974 *Bull. Inst. Math. Appl.* **10** 266
[12] Levine D and Steinhardt P J 1984 *Phys. Rev. Lett.* **53** 2477
[13] Jeong H-C and Steinhardt P J 1997 *Phys. Rev.* B **55** 3520
[14] Steinhardt P J and Jeong H-C 1996 *Nature* **362** 431
[15] Rabson D A, Rokhsar D S, Wright D C and Mermin N D 1991 *Rev. Mod. Phys.* **63** 699
[16] Grünbaum B and Shephard G 1989 *Tilings and Patterns* (New York: Freeman)
[17] Bellissard J 1990 Renormalization group analysis and quasicrystals *Preprint* CPT-90-PE-2484
[18] Ercolessi E, Landi G and Teotonio-Sobrinho P 1998 *Rev. Math. Phys.* **10** 439
(Ercolessi E, Landi G and Teotonio-Sobrinho P 1996 *Preprint* q-alg/9607017)
[19] Baake M, Hermisson J and Pleasants P 1997 *J. Phys. A: Math. Gen.* **30** 3029
[20] Blackadar B 1986 *K-Theory for Operator Algebras; MSRI Publ.* vol 5 (Berlin: Springer)
[21] Jones V F R 1983 *Invent. Math.* **72** 1–25
[22] Goodman F, de la Harpe J and Jones V F R 1989 *Coxeter Graphs and Towers of Algebras; MSRI Publ.* vol 14 (Berlin: Springer)
[23] Coquereaux R 1995 *Adv. Appl. Math.* **16** 402
[24] Wegge-Olsen N 1993 *K-theory and C^*-Algebras—a Friendly Approach* (Oxford: Oxford Science)
[25] Lawson B 1974 *Bull. Am. Math. Soc.* **80** 369
[26] Camacho C and Neto A 1985 *Geometric Theory of Foliations* (Boston: Birkhauser)
[27] Pinsner M and Voiculescu D 1980 *J. Oper. Theory* **4** 201
[28] Landi G, Lizzi F and Szabo R J 1999 From large N matrices to the noncommutative torus *Preprint* hep-th/9912130
[29] Janot C 1992 *Quasicrystals—A Primer* (Oxford: Clarendon Press)
[30] Olds C D 1963 *Continued Fractions* (New York: Mathematical Association of America)
[31] Bigatti D 1998 Noncommutative geometry for outsiders *Preprint* hep-th/9802129
[32] Onoda G Y, Steinhardt P J, Divincenzo D P and Socolar J E S 1988 *Phys. Rev. Lett.* **60** 2653
[33] de Bruijn N G 1981 *Nederl. Akad. Wetensch. Proc.* A **84** 39